

Bounds on the slope and curvature of Isgur-Wise function in a QCD inspired quark model

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Abstract

The QCD inspired potential model pursued by us earlier has been recently modified to incorporate an additional factor ' c ' in the linear cum Coulomb potential. While it facilitates the inclusion of standard confinement parameter $b = 0.183 GeV^2$, unlike in previous work, it still falls short of explaining the Isgur-Wise function for the B mesons without adhoc adjustment of the strong coupling constant .

In this work, we determine the factor ' c ' from the experimental values of decay constants and masses and show that the reality constraint on ' c ' yields bounds on the strong coupling constant as well on slope and curvature of Isgur-Wise function allowing more flexibility to the model.

Keywords: Dalgarno method ,Isgur-Wise function,slope, curvature.

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1 Introduction

In recent years, considerable experimental and theoretical efforts have been undertaken to understand the physics of hadrons containing a heavy quark [1]. The Isgur-Wise function [2] is an important quantity in this area of hadron

physics. It is in this spirit, that this function has been studied in various quark models [3]–[12] besides QCD sum rule approach [13], the MIT bag model [14] and the Skryme model [15].

Since one of the basic ingredients of the IW function is the hadron wavefunction involving heavy quark [3]–[12], it is therefore meaningful to test any specific QCD inspired quark model by calculating the IW function and study it phenomenologically.

Sometimes back, a specific QCD inspired quark model was proposed by us [16] which had later been used to calculate IW function as well [17, 18, 19]. One of drawback of the model is that significant confinement effect could not be accomodated in the model [16, 17, 18] due to perturbative constraints coming from using the Dalgarno's method [20]. Only recently [19], standard confinement effect $b = 0.183 GeV^2$ [21] was accomodated in the improved version of QCD inspired quark model brought through the introduction of parameter ' c ' in the potential : $V = \frac{-4\alpha_s}{3r} + br + c$ taking $c \sim 1 GeV$ as its natural scale fixing and $A_0 = 1$ where A_0 is an undetermined factor appearing in the series solution of the Schrödinger equation [Eq.(8) of Ref.19]. In earlier works [16, 17, 18], the unknown coefficient cA_0 occurred in the wavefunction was set to zero.

One of the drawback of work [19] was the adhoc enhancement of strong coupling constant was needed to take into account of the slope and curvature of B, B_s and B_c mesons.

In this work, we take an alternative strategy to remove this adhoc enhancement. We use the wavefunction at the origin involving the unknown coefficient cA_0 and fix it from the experimental values of masses and decay constants directly. The reality constraint on cA_0 will then yeild lower bounds on the strong coupling constant α_s , which would lead to the upper bounds on the slope and curvature of the IW function.

The rest of the paper is organised as follows : Section 2 contains the theory of the improved QCD inspired quark model, Section 3 encloses the results and in Section 4 we draw conclusion and remarks.

2 Theory

2.1 The Wavefunction

The spin independent Fermi-Breit Hamiltonian for ground state ($l = 0$), neglecting the contact term proportional to δ^3 is [16, 17] :

$$\begin{aligned} H &= H_o + H' \\ &= -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r} + br + c \end{aligned} \quad (1)$$

where α_s is the running coupling constant , b is the confinement parameter and c is another parameter whose significance will be cleared later.

In this work, the α_s values are taken from the V -scheme [18, 26, 27] as done in [19] which are large as compared to those of \overline{MS} -scheme. It is necessary as large α_s values lead to better results for slope and curvature of Isgur-Wise function in earlier work [17, 18, 19].

In earlier work [17, 18] , large confinement was found to be inconvenient in the calculation of slope and curvature of Isgur -Wise function which was a limitation of the model. So, our aim has been focussed in the inclusion of larger confinement and hence we retain the same choice of $b = 0.183 GeV^2$ [19, 21] to investigate whether this approach leads to better results or not. It is worth notable that inclusion of c in the analysis [19] allows larger b useful , so we don't think another choice of b .

With , $H_o = -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r}$ as the parent Hamiltonian and $H' = br + c$ as the perturbed Hamiltonian , we obtain a ground state wavefunction upto the first order correction using the Dalgarno method [20] of stationary state perturbation theory as :

$$\psi_{conf}(r) = N \left(cA_0 + \frac{1}{\sqrt{\pi a_0^3}} - \frac{\mu b a_0 r^2}{\sqrt{\pi a_0^3}} \right) e^{-\frac{r}{a_0}} \quad (2)$$

where A_0 is the unknown coefficient appearing in the series solution of the Dalgarno method

Including the relativistic effect [22, 23], the wavefunction is :

$$\psi_{conf+rel}(r) = N' \left(cA_0 + \frac{1}{\sqrt{\pi a_0^3}} - \frac{\mu b a_0 r^2}{\sqrt{\pi a_0^3}} \right) \left(\frac{r}{a_0} \right)^{-\epsilon} e^{-\frac{r}{a_0}} \quad (3)$$

Here a_0 is given by:

$$a_0 = \frac{3}{4\mu\alpha_s} \quad (4)$$

and

$$\epsilon = 1 - \sqrt{1 - \frac{4\alpha_s}{3}} \quad (5)$$

N and N' are the normalization constants given by :

$$N^2 = \frac{1}{1 + \frac{45\mu^2 b^2 a_0^6}{8} - 3\mu b a_0^3 + \pi a_0^3 c^2 A_0^2 + \frac{2cA_0\pi a_0^3}{\sqrt{\pi a_0^3}} - \frac{3\pi a_0^6 c A_0 \mu b}{\sqrt{\pi a_0^3}}} \quad (6)$$

and

$$N'^2 = \frac{2^{7-2\epsilon}}{\Gamma(3-2\epsilon) X_1} \quad (7)$$

where X_1 is given in APPENDIX A.

We note that the equations (2),(3),(6)and (7) are obtained from Eq.(4),(6),(5) and (7) of Ref.[19] exhibiting explicit dependence of cA_0 in them.

2.2 Fixing of the coefficient cA_0

The wavefunction at the origin (WFO), is related to the decay constant f_p and the mass of the pseudoscalar meson M_p through the relation [16, 24]:

$$|\psi(0)|^2 = \frac{f_p^2 M_p}{12} \quad (8)$$

Again from equation (2), we have :

$$|\psi(0)|^2 = N^2 \left[c^2 A_0^2 + \frac{1}{\pi a_0^3} + \frac{2cA_0}{\sqrt{\pi a_0^3}} \right] \quad (9)$$

Using equation(6), we arrive at the quadratic equation for cA_0 :

$$A' (cA_0)^2 + B' (cA_0) + C' = 0 \quad (10)$$

where

$$A' = \pi a_0^3 |\psi(0)|^2 - 1 \quad (11)$$

$$B' = 2\sqrt{\pi a_0^3} |\psi(0)|^2 - 3\mu b a_0^3 \sqrt{\pi a_0^3} |\psi(0)|^2 \quad (12)$$

and

$$C' = |\psi(0)|^2 \left[1 + \frac{45\mu^2 b^2 a_0^6}{8} - 3\mu b a_0^3 \right] - \frac{1}{\pi a_0^3} \quad (13)$$

Using the experimental values of f_p and M_p [25], we determine $|\psi(0)|^2$ from equation(9) which in turn will yield two solutions for cA_0 in equation (10):

$$cA_0 = \frac{-B' \pm \sqrt{B'^2 - 4A'C'}}{2A'} \quad (14)$$

which will depend on μ, M_P, f_P and α_s . The solution corresponding to the +ve(-ve) sign of equation(15) will be termed as +ve(-ve) solution hereafter. It will be shown numerically that for a given μ, M_P , and f_P , α_s reaches the minimum value when the following condition is satisfied :

$$B'^2 - 4A'C' = 0 \quad (15)$$

The formalism involving Eq.(5)-(16) is strictly valid only without relativistic effect as the wavefunction at the origin with such effect [Eq.(3)] is not well defined due to its singularity at the origin. For a subsequent analysis, we assume that cA_0 does not deviate significantly from its non-relativistic value so that it can be used to calculate the slope and curvature of the IW function even without relativistic effect.

2.3 Charge radius (slope) and convexity parameter (curvature) of I-W function

The Isgur-Wise function is written as [2, 17] :

$$\begin{aligned}\xi(v_\mu \cdot v'_\mu) &= \xi(y) \\ &= 1 - \rho^2 (y - 1) + C (y - 1)^2 + \dots\end{aligned}\quad (16)$$

where

$$y = v_\mu \cdot v'_\mu \quad (17)$$

and v_μ and v'_μ being the four velocity of the heavy meson before and after the decay. The quantity ρ^2 is the slope of I-W function at $y = 1$ and known as charge radius :

$$\rho^2 = \left. \frac{\partial \xi}{\partial y} \right|_{y=1} \quad (18)$$

The second order derivative is the curvature of the I-W function known as convexity parameter :

$$C = \frac{1}{2} \left[\left. \frac{\partial^2 \xi}{\partial^2 y} \right|_{y=1} \right] \quad (19)$$

For the heavy-light flavor mesons the I-W function can also be written as [6, 17] :

$$\xi(y) = \int_0^{+\infty} 4\pi r^2 |\psi(r)|^2 \cos pr dr \quad (20)$$

where

$$p^2 = 2\mu (y - 1) \quad (21)$$

Equation (21) holds good for both relativistic and nonrelativistic case. The wavefunction $\psi(r)$ takes different form for both the cases. Without relativistic effect, it is given by equation(2) and with relativistic effect it is given by (3).

With the wavefunction(2) in equation(10) i.e. including confinement only the charge radius ρ_{conf}^2 and convexity parameter C_{conf} are respectively given

by:

$$\rho_{conf}^2 = \frac{\mu^2[24\pi c^2 A_0^2 a_0^5 + 24a_0^2 + 630\mu^2 b^2 a_0^8 + 48cA_0\sqrt{\pi a_0^7} - 180cA_0\mu b\sqrt{\pi a_0^{13}} - 180\mu b a_0^5]}{8\pi c^2 A_0^2 a_0^3 + 8 + 45\mu^2 b^2 a_0^6 + 16cA_0\sqrt{\pi a_0^3} - 24\mu bcA_0\sqrt{\pi a_0^3} - 24\mu b a_0^3} \quad (22)$$

and :

$$C_{conf} = \frac{\mu^4[60\pi c^2 A_0^2 a_0^7 + 60a_0^4 + 4725\mu^2 b^2 a_0^{10} + 120cA_0\sqrt{\pi a_0^{10}} - 840cA_0\mu b\sqrt{\pi a_0^{17}} - 840\mu b a_0^7]}{16\pi c^2 A_0^2 a_0^3 + 16 + 90\mu^2 b^2 a_0^6 + 32cA_0\sqrt{\pi a_0^3} - 48\mu bcA_0\sqrt{\pi a_0^3} - 48\mu b a_0^3} \quad (23)$$

With the wavefunction (3) in equation (10) i.e. including both relativistic and confinement effect the charge radius $\rho_{conf+rel}^2$ and convexity parameter $C_{conf+rel}$ are given by :

$$\rho_{conf+rel}^2 = \frac{\mu^2 a_0^2 (4 - 2\epsilon) (3 - 2\epsilon) [X_1]}{4[X_2]} \quad (24)$$

and

$$C_{conf+rel} = \frac{\mu^4 a_0^4 (6 - 2\epsilon) (5 - 2\epsilon) (4 - 2\epsilon) (3 - 2\epsilon) [X_3]}{96[X_2]} \quad (25)$$

where X_1, X_2 and X_3 are given in Appendix.

We note that equations (25) and (26) are equivalent to equations (18) and (19) of Ref[19] exhibiting explicit cA_0 dependence.

3 Results

3.1 Values of cA_0 and lower bounds on α_s

As noted earlier, cA_0 depends on μ, M_P, f_P and α_s . In fig.1(a-e) we plot cA_0 vs α_s for D, D_s, B, B_s and B_c mesons. It shows that α_s tends to reach the minimum value when two solutions of Eq.(11) almost merge satisfying the condition (16). This feature is true for any set of the parameters μ, f_p and M_p . In table 1, we give the lower bounds on α_s for mesons having c and b quarks.

The dependence of cA_0 on α_s and μ can be noted as follows :
With constant μ , cA_0 decreases with α_s values rising and vice-versa. On the

Table 1: Lower Bounds on α_s

Mesons	Quark content	$\mu(\text{GeV})$ Ref[25]	$M_p(\text{GeV})$ Ref[25]	$f_p(\text{GeV})$ Ref[25]	cA_0	Lower bound on α_s
D	$c\bar{u}/cd$	0.276	1.869	0.192	0.9665	~ 0.601
B	$b\bar{u}/bd$	0.315	1.968	0.157	0.7653	~ 0.652
D_s	$c\bar{s}$	0.368	5.279	0.210	1.1967	~ 0.49
B_s	$b\bar{s}$	0.44	5.279	0.171	0.999	~ 0.493
B_c	bc	1.18	5.37	0.36	1.167	~ 0.302

other hand, with constant α_s , cA_0 increases(decreases) with increase (decrease) in μ .

3.2 Bounds on slope and curvature of the IW function

Using the lower bounds on α_s for each heavy-light and heavy-heavy mesons, we obtain upper bounds on the slope and curvature of the I-W function using equations (23),(24),(25) and (26). They are listed in table 2. We note that with increasing α_s values, the slope and curvature decreases and henceforth the lower bound on α_s corresponds to the upper bound on ρ^2 and C .

In table 3, we record the predictions of the slope and curvature of the IW function in various models while in table 4, we reproduce the corresponding predictions of the model of Ref.[19] with $c = 1\text{GeV}$ and $A_0 = 1$ in V-scheme [26, 27] for various mesons. Two values for B , B_s and B_c mesons are shown where case- a) represents the actual values for ρ^2 and C in that work with $\alpha_s = 0.261$; while case-b) represents those for adhoc adjustable value of $\alpha_s = 0.60$ in order to show the usefulness of large α_s as mentioned in Ref 19. The α_s are already large for D and D_s mesons, so no two values are shown.

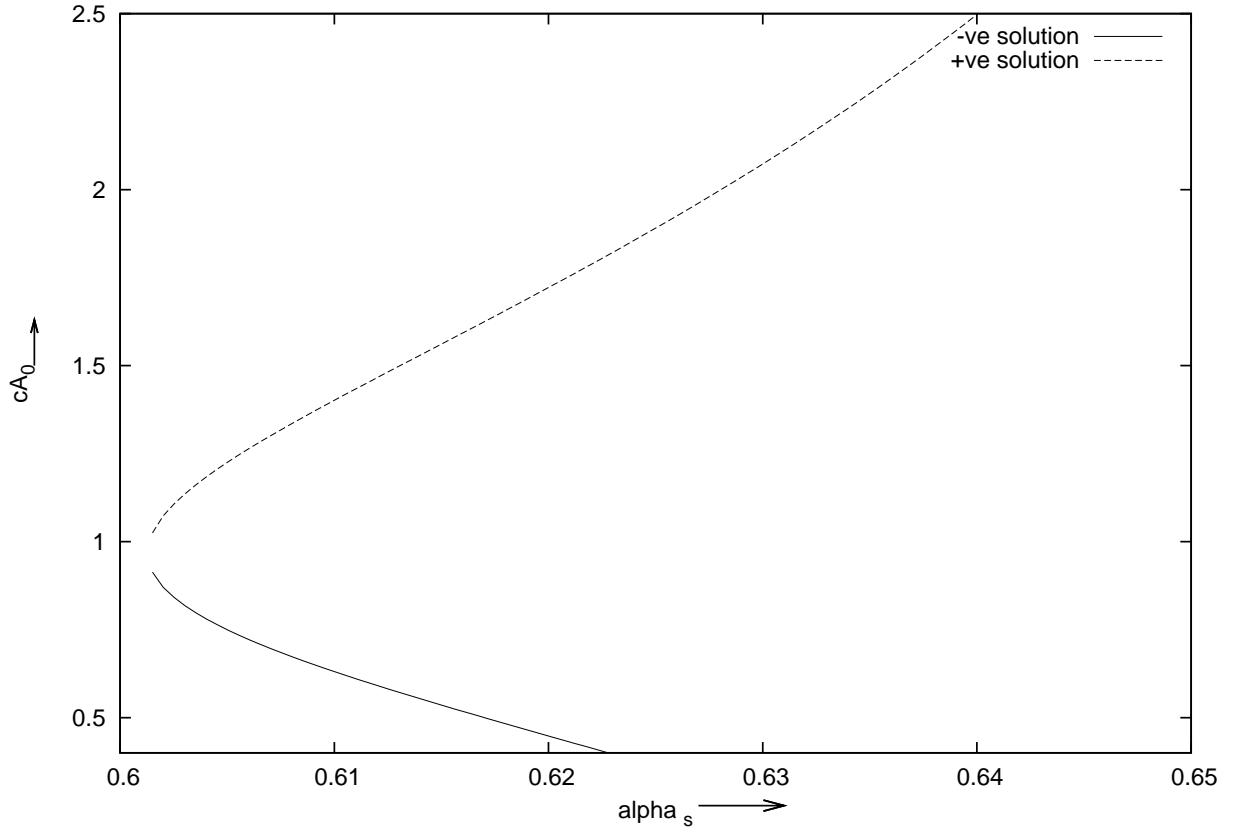


Figure 1: Variation of cA_0 vs α_s for D Meson. The +ve (-ve) solution of Eq.15 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim 0.601$, the lower bound on α_s corresponding to the solution of Eq.16 for D Meson.

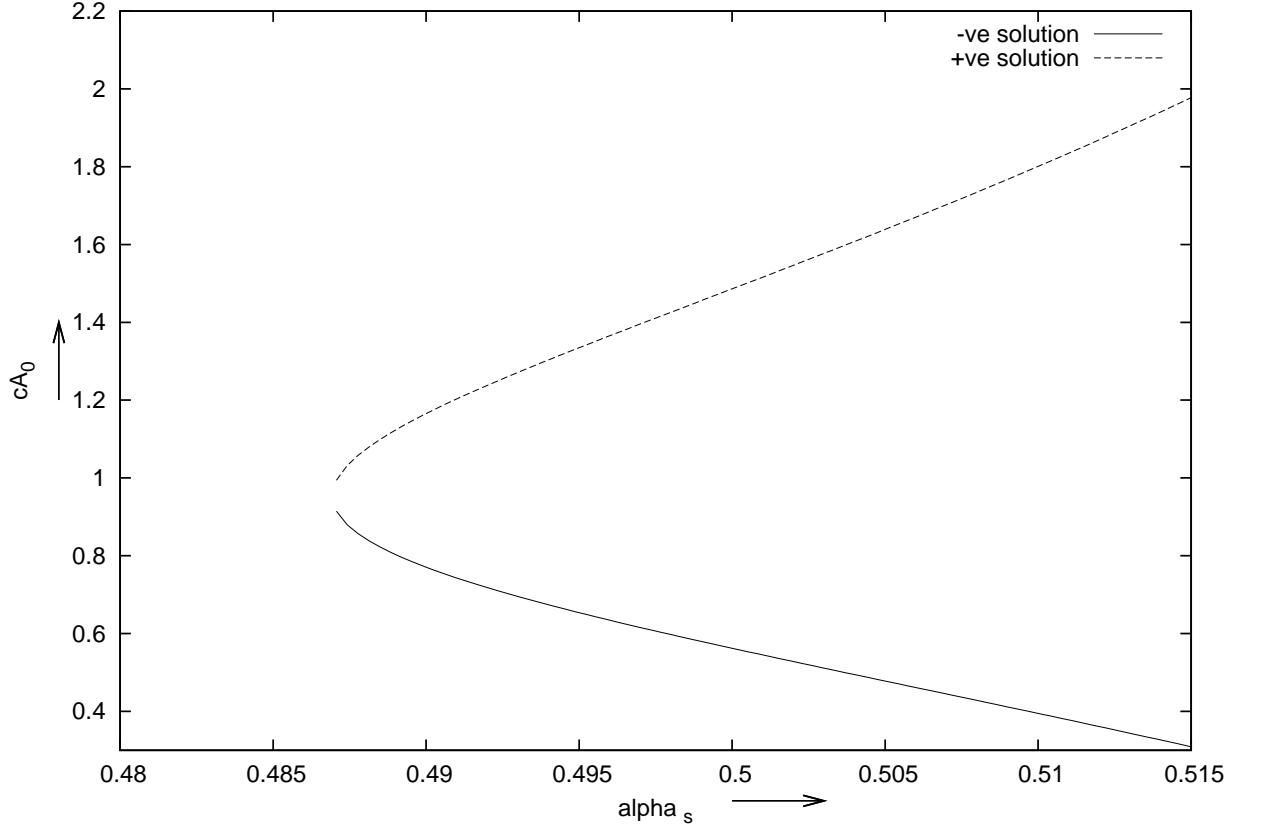


Figure 2: Variation of cA_0 vs α_s for D_s Meson. The +ve (-ve) solution of Eq.15 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim 0.49$, the lower bound on α_s corresponding to the solution of Eq.16 for D_s Meson.

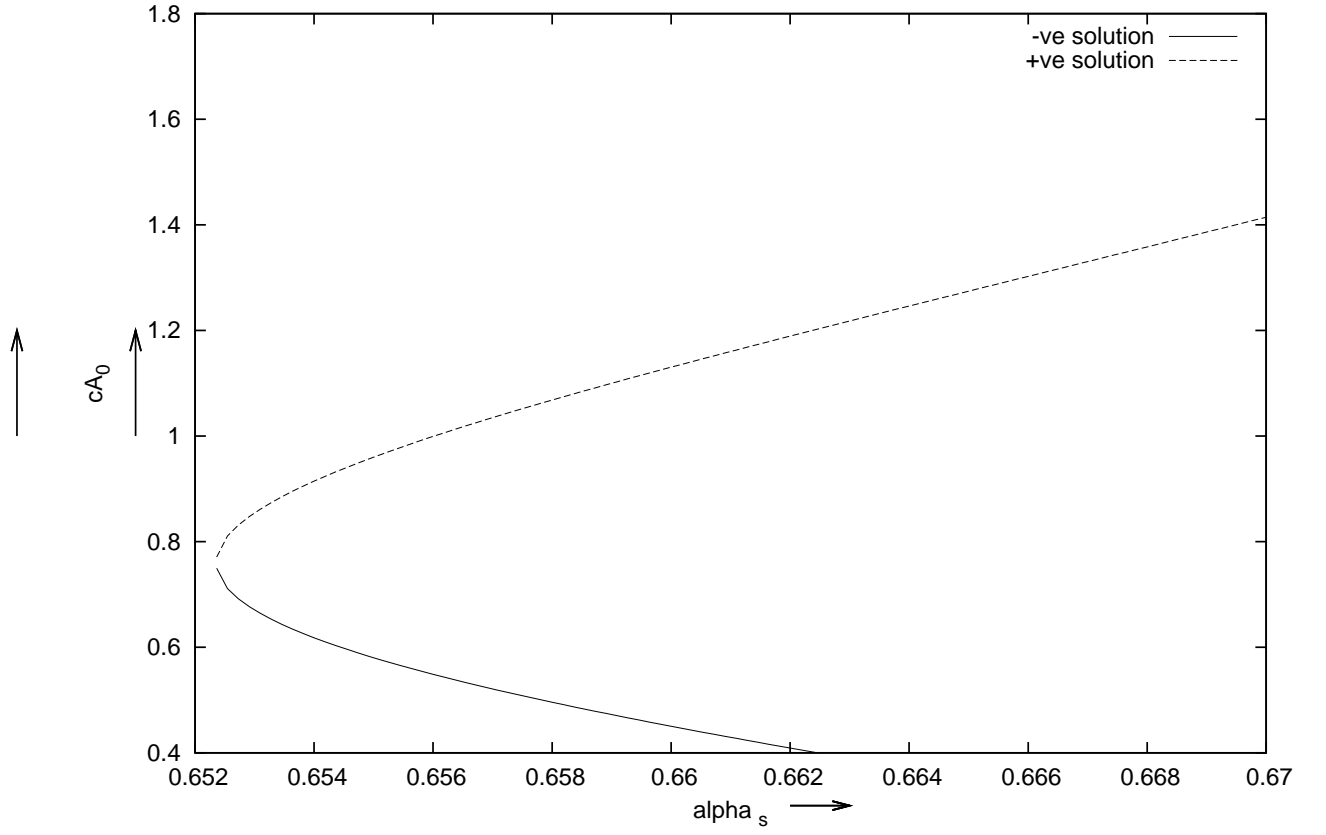


Figure 3: Variation of cA_0 vs α_s for B Meson. The +ve (-ve) solution of Eq.15 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim 0.652$, the lower bound on α_s corresponding to the solution of Eq.16 for B Meson.

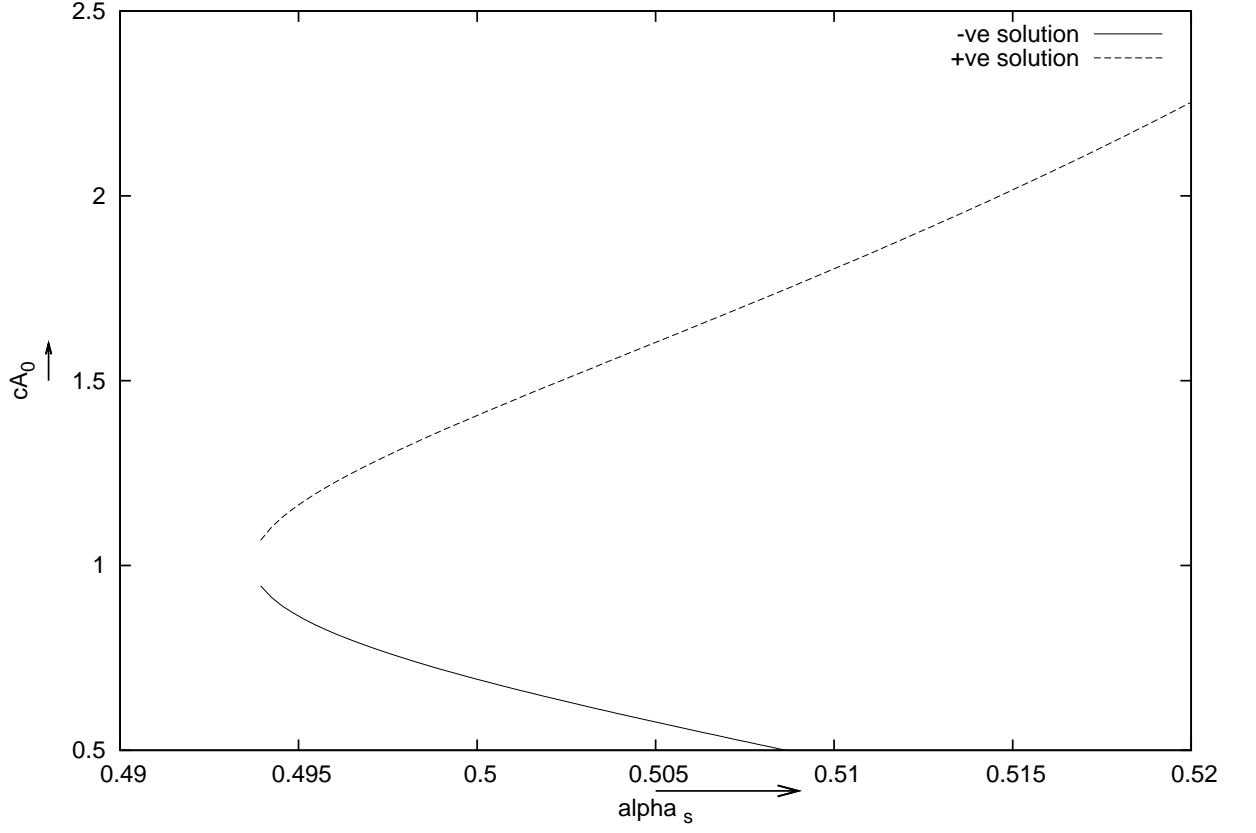


Figure 4: Variation of cA_0 vs α_s for B_s Meson. The +ve (-ve) solution of Eq.15 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim 0.493$, the lower bound on α_s corresponding to the solution of Eq.16 for B_s Meson.

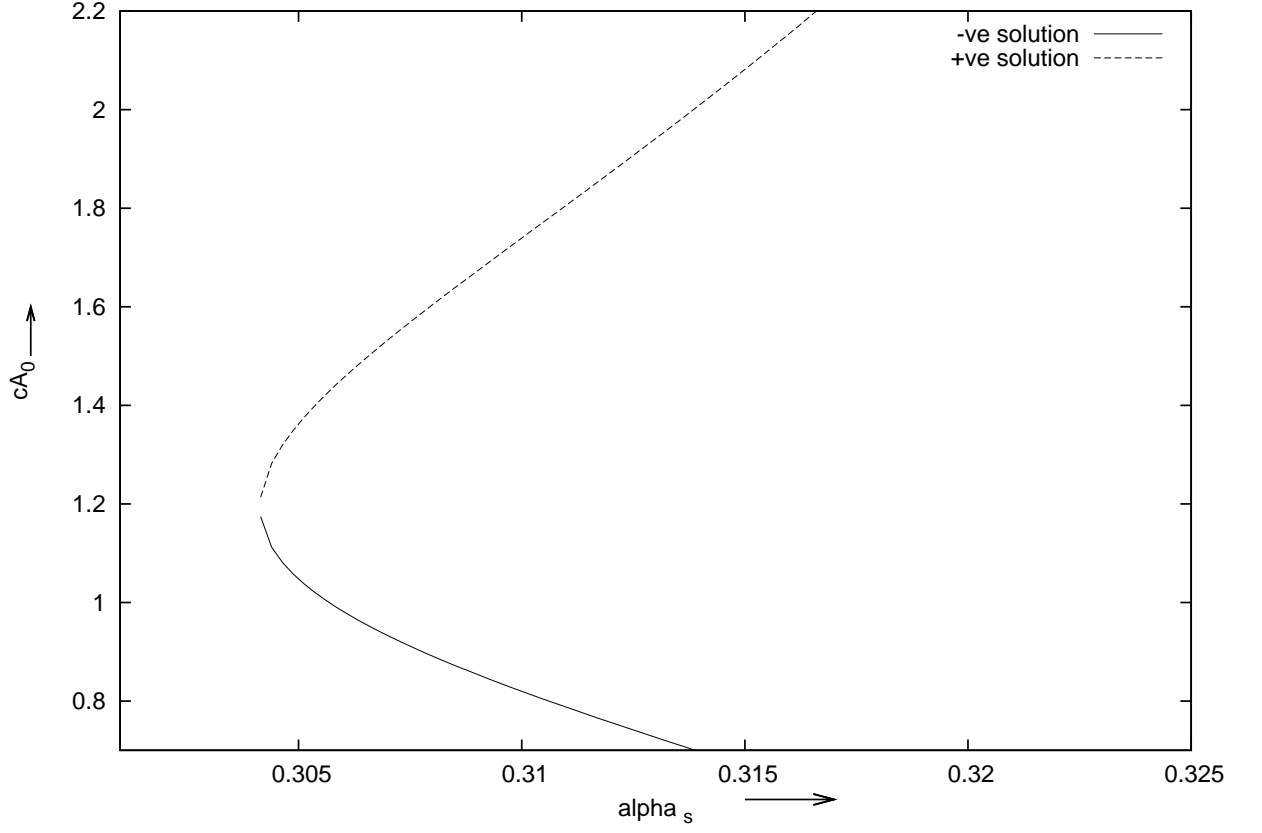


Figure 5: Variation of cA_0 vs α_s for B_c Meson. The +ve (-ve) solution of Eq.15 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim 0.302$, the lower bound on α_s corresponding to the solution of Eq.16 for B_c Meson.

Table 2: Upper Bounds on Slope and Curvature

Meson (Quark Content)	Slope ρ^2		Curvature C	
	Without relati- vistic effect	With relati- vistic effect	Without relati- vistic effect	With relati- vistic effect
$D(c\bar{u}/cd)$	6.78	1.675	13.19	5.138
$B(bu/bd)$	5.78	1.016	9.58	1.29
$D_s(c\bar{s})$	9.115	3.067	26.48	14.32
$B_s(bs)$	11.92	2.652	34.49	6.902
$B_c(bc)$	28.46	10.39	219.46	45.23

Table 3: Predictions of the slope and curvature of the IW function in various models.

Model	Value of ρ^2	Value of curvature C
Yaouanc et al [28]	≥ 0.75	..
Yaouanc et al [12]	≥ 0.75	≥ 0.47
Rosner et al [29]	1.66	2.76
Mannel et al[30, 31]	0.98	0.98
Pole Ansatz [32]	1.42	2.71
MIT Bag Model [14]	2.35	3.95
Simple Quark Model [3]	1	1.11
Skryme Model [15]	1.3	0.85
QCD Sum Rule [13]	0.65	0.47
Relativistic Three Quark Model [4]	1.35	1.75
Infinite Momentum Frame Quark Model [5]	3.04	6.81

Table 4: Predictions of the slope and curvature of the I-W function in the QCD inspired quark model according to Ref[19] with $c = 1$ and $A_0 = 1$ taking relativistic and confinement effect in V-scheme. This table is nothing , but the copy of the last rows of tables 1,2,3 of Ref 19 .

Meson	α_s	slope (ρ^2)	curvature(C)
D	0.625	1.136	5.377
D_s	0.625	1.083	3.583
B	a)0.261 b)0.60	a)128.128 b)1.329	a)5212 b)7.2
B_s	a)0.261 b)0.60	a)112.759 b)1.257	a)4841 b)4.379
B_c	a)0.261 b)0.60	a)44.479 b)1.523	a)2318 b)0.432

4 Conclusion and Remarks

In this paper, we have shown that the reality bound on cA_0 puts lower limit on α_s and correspondingly upper limit on ρ^2 and C .

Furthermore, with cA_0 , the upper bounds on ρ^2 and C decrease which is evident from the above list of bounds[table-2]. The estimated upper bounds on ρ^2 and C for all the mesons are found to be consistent with other models and data [table-3] without making any adhoc enhancement of the strong coupling constant as had been done in ref (19)[table-4]. From the phenomenological point of view we note that in the nonrelativistic limit, the universal form factor and Isgur-Wise function for semileptonic decay $B \rightarrow D^* l \nu$ are identical when subleading terms in velocity and terms of order $O\left(\frac{E_b}{m_Q}\right)$ are neglected with E_b as the binding energy and m_Q as the mass of heavy quark [33]. However even if we make calculation for the universal form factor for finite mass, we obtain to first order in $(y - 1)$ as $0.8 - 2.57(y - 1)$ which seems to be satisfactory [33, 34].

It is worth notable that in the limit $cA_0 \rightarrow 0$, there will be no bounds on α_s as well as on ρ^2 and C ; rather fixed values of α_s have to be used to get definite set of ρ^2 and C . So, in that case, the analysis will turn to that of ref [17,18] where large confinement could not be (i.e. $b = 0.183 \text{ GeV}^2$) incor-

porated e.g. tables -(1,3) of ref[17] and tables -(2,3) of ref[18].

We conclude this paper with a comment on the physical significance of the factor ' c ' that has become so crucial for our analysis of bounds on slope and curvature.

It is common wisdom that a constant potential like ' c ' just scales the energies and doesnot affect the wavefunction nor does it change physics.This can be seen from the hydrogen atom problem with the potential $V(r) == \frac{-A}{r} + c$.However , if one uses ' c ' as the perturbation instead of as parent in the Dalgarno method of perturbation theory [20],the wavefunction for the H -atom becomes:

$$\psi(r) = N_1 \left(cA_0 + \frac{1}{\sqrt{\pi a_0^3}} \right) e^{-\frac{r}{a_0}}$$

to be compared with the wavefunction with ' c ' as parent:

$$\psi(r) = \left(\frac{1}{\sqrt{\pi a_0^3}} \right) e^{-\frac{r}{a_0}}$$

where the normalization constant:

$$N_1^2 = \frac{1}{1 + \pi a_0^3 c^2 A_0^2 + \frac{2cA_0\pi a_0^3}{\sqrt{\pi a_0^3}}}$$

Thus, the perturbative child ' c ' rather than the parent ' c ' plays the crucial role in the present analysis.

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X_1 , X_2 and X_3 are evaluated as :

$$\begin{aligned} X_1 = & 64\pi c^2 A_0^2 a_0^3 + 64 + \mu^2 b^2 a_0^6 (8 - 2\epsilon) (7 - 2\epsilon) (6 - 2\epsilon) (5 - 2\epsilon) \\ & + 128cA_0\sqrt{\pi a_0^3} - 16cA_0\mu b\sqrt{\pi a_0^9} (6 - 2\epsilon) (5 - 2\epsilon) \\ & - 16\mu b a_0^3 (6 - 2\epsilon) (5 - 2\epsilon) \end{aligned} \quad (26)$$

$$\begin{aligned} X_2 = & 64\pi c^2 A_0^2 a_0^3 + 64 + \mu^2 b^2 a_0^6 (6 - 2\epsilon) (5 - 2\epsilon) (4 - 2\epsilon) (3 - 2\epsilon) \\ & + 128cA_0\sqrt{\pi a_0^3} - 16cA_0\mu b\sqrt{\pi a_0^9} (4 - 2\epsilon) (3 - 2\epsilon) \\ & - 16\mu b a_0^3 (4 - 2\epsilon) (3 - 2\epsilon) \end{aligned} \quad (27)$$

$$\begin{aligned} X_3 = & 64\pi c^2 A_0^2 a_0^3 + 64 + \mu^2 b^2 a_0^6 (10 - 2\epsilon) (9 - 2\epsilon) (8 - 2\epsilon) (7 - 2\epsilon) \\ & + 128cA_0\sqrt{\pi a_0^3} - 16cA_0\mu b\sqrt{\pi a_0^9} (8 - 2\epsilon) (7 - 2\epsilon) \\ & - 16\mu b a_0^3 (8 - 2\epsilon) (7 - 2\epsilon) \end{aligned} \quad (28)$$

Not only the above expressions ,but all the integrals in the analysis are evaluated with the help of Gamma function given by :

$$\frac{\Gamma(n+1)}{\alpha^{n+1}} = \int_0^{+\infty} r^n e^{-\alpha r} dr \quad (29)$$